A note on the respective widths of confidence intervals for means with known and unknown variances

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Résumé

Most professors of elementary statistics assume implicitly that confidence intervals for means of normal distributions with unknown variances are wider than the ones where the variances are assumed known. The reason being that less knowledge brings more uncertainty, that entails in turn wider confidence intervals. This is not always the case. This simple fact follows from elementary considerations.

1 Introduction

Most, if not all, textbooks describe the two usual confidence intervals (CIs) for the mean of a normal random variable X : the first one in the case where the variance of X is known and the other where the variance is not known. Most professors of elementary statistics would implicitly assume that the first one has a smaller width since the width of the interval is related to the knowledge concerning the variable X : less knowledge brings more uncertainty, thus, for a given level of confidence, a wider confidence interval for means. The width of CIs is clearly related to the uncertainty concerning values of the variable. Is σ is large, more uncertainty is entailed for the location

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of the mean of X. And the answer to a question to the effect of the absence of the exact value of the standard deviation of X on the CIS, would most probably bring an answer along these lines.

Even practically oriented textbooks (e.g. Ramsey & Shaffer, 2002 ; Devore, 2000 ; to cite only a few) do not touch upon the question, the development concentrates instead on the large or small sample determination of the CIs, and the use of sample standard deviations. They sometimes describe the frequentist interpretation for the covering of the real value of the mean by the CIs.

In reality, because of the skewness of the χ^2 distribution related to S and σ, it is easy to show that, especially for small samples, a good proportion of the CIs using s instead of σ have a smaller width.

This fact became obvious when the author used the animation he has designed to visually provide the students with a comprehension of the frequentist interpretation of CIs for means, also mentioned by the aforementioned authors and many others. Proof, if need there be, that Information Technologies can be very efficient for the fast and precise understanding of some statistical concepts, and CIs are surely one of them[1].

2 The widths of the confidence intervals for means

If one assumes that the distribution of a given random variable, X, is normal, there are two common CIs, with confidence $1-\alpha$, for the mean of X obtained from n independant samples $x_i, i = 1, \ldots, n$, of X :

$$\overline{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}} , \tag{1}$$

when the variance of X is assumed known, and when this is not the case, one uses the sample standard deviation for the n samples :

$$\overline{x} \pm t_{n-1;\alpha/2} \frac{s}{\sqrt{n}} . \tag{2}$$

The values of $z_{\alpha/2}$, and $t_{n-1;\alpha/2}$ are respectively the $1-\alpha/2$ quantiles for the standard normal and the Student's T_{n-1} distributions.

[1] One can find this Flash animation and some others on the page http ://magi.polymtl.ca/bourdeau/Mth2301/index.html.

The first one is wider than the second if and only if :

$$\frac{s}{\sigma} \leq \frac{z_{\alpha/2}}{t_{n-1;\alpha/2}}\,.$$

Since $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$, we easily get the probability p that (1) is wider than (2) :

$$p = \mathrm{P}[\,\chi^2_{n-1} \leq (n-1)K\,]\,, \tag{3}$$

where we have put $K \equiv (z_{\alpha/2}/t_{n-1;\alpha/2})^2$.

Furthermore, since T_{n-1} converges to Z, the standard normal, when $n \uparrow \infty$, and that the quantiles of T_{n-1} are larger than those of Z for all n, K is smaller than 1 and one can check that, for any confidence level $(1-\alpha)$, it increases to 1 with $n \uparrow \infty$. As a consequence, it can be checked that for any confidence level the $(n-1)K$ in (3) is smaller than the means of the χ^2_{n-1}, but grows to be their means when n increases indefinitely.

One can also note that, as n gets larger and larger, the χ^2_{n-1}'s with means $n-1$ approach normal random variables, and thus the probabilities p in (3) tend to $\frac{1}{2}$. In the examples obtained through the animation mentioned earlier, this fact is masked because the two confidence intervals are then almost equal.

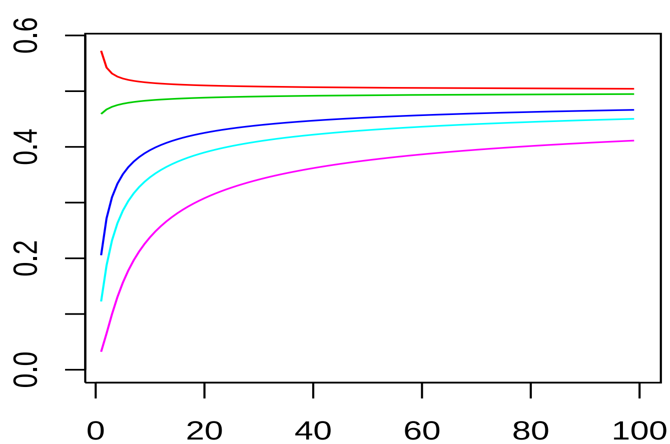

FIG. 1 – Graphs of the probabilities p against sample sizes from $n = 2$ to $n = 100$. Five cases with different confidence levels, from the bottom to the last upper graphic : $1-\alpha = 0.99, 0.95, 0.90, 0.60, 0.10$.

Although the curves in Figure 1 appear to be monotonous, the situation is a bit different for confidence levels in the vicinity of .5. Figure 2 presents similar curves for small sample sizes, with confidence levels from .46 to .56 with steps of .02.

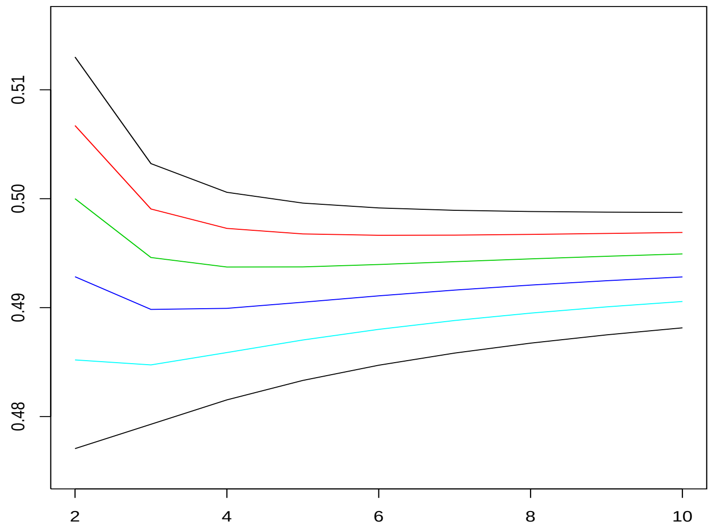

Fig. 2 – Graphs of the probabilities p against sample sizes from $n = 2$ to $n = 10$. Five cases with different confidence levels, from the bootm to the last upper graphic : $1 - \alpha = 0.56$, to $1 - \alpha = .46$ by steps of .02.

From Figure 1, we see that the probability p increases with decreasing confidence levels. For sample sizes less than 10, and very high confidence levels, p is still in the order of 10%. With the "usual" 95% confidence level, and sizes in the order of a few dozens, p is close to 40%.

3 Conclusion

A simple calculation has shown that, contrary to intuition, it is true that a fairly large proportion of the confidence intervals for means of normal random variables with the assumption that the standard deviation is unknown are smaller than the ones calculated assuming a known standard deviation. When the sample size is large, and the confidence small, this proportion gets close to one half. For all practical purposes, with sample sizes around 20, this proportion goes from roughly 30% for a confidence level of 99% to slightly more than 40% for a confidence level of 90%. These proportions increase with n.

It is interesting to note that, in spite of the fact that the the calculation to find the answer extremely simple, this question, though very natural, was brought to light by a computer animation that was designed to facilitate comprehension of the concept of confidence intervals for students of elementary statistics.

Références

[1] Devore, J. L. (2000), *Probability and Statistics for Engineers and Scientists,* 5th edition, Pacific Grove CA : Duxbury.

[2] Ramsey, F. L., Schafer, D.W. (2002), *The Statistical Sleuth. A Course in Methods of Data Analysis,* 2nd edition, Pacific Grove CA : Duxbury.